On the concept of energy in classical relativistic physics

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Abstract. The relativistic concept of energy is discussed under two important aspects: the behaviour of the energy under coordinate transformations and its general definition in the presence of external gravitational and electromagnetic fields. On the basis of energy conservation and a non-relativistic approximation it is argued that only the zeroth component of the covariant 'generalized' momentum should be called 'energy'.

Zusammenfassung. Das relativistische Energiekonzept wird unter zwei wichtigen Aspekten diskutiert: Das Verhalten der Energie unter Koordinatentransformationen und ihre allgemeine Definition in Anwesenheit eines äußeren Gravitations- und elektromagnetischen Feldes. Aufgrund von Energieerhaltung und der nichtrelativistischen Approximation wird argumentiert, daß nur die nullte Komponente des kovarianten 'verallgemeinerten' Impulses as 'Energie' bezeichnet werden sollte.

1. Introduction

The concepts of mass and energy in classical relativistic physics, symbolized by Einstein's relation $E = mc^2$, are still discussed controversially in their historical, pedagogical and theoretical aspects, as can be seen from the large number of recent contributions (Adler 1987, Feigenbaum and Mermin 1988, Fadner 1988, Okun 1989, Rindler 1990, Strnad 1991). The purpose of this article is to remark on two aspects of the concept of energy that so far have not been fully estimated in the author's opinion. The first aspect is the behaviour under coordinate transformation; here we conclude that the energy should be defined as the zeroth component of the covariant vector of the generalized momentum. The second is the energy of a particle in a gravitational field; here we arrive at the general conclusion $E \neq mc^2$.

2. The theoretical basis

Okun (1989) already notes that the mass (he means only the rest mass or proper mass) is a relativistic invariant, whereas the energy transforms as a component of a 4-vector. Hence we can have $E_0 = mc^2$ as a relation valid only in the rest system of a body. We look at this as an example of the general principle that any physical quantity is well defined only if its transformation properties under coordinate changes are specified. For relativistic physics this means that we have to specify whether a quantity is a scalar, a component of a covariant or contravariant vector, a tensor of certain rank etc. We will therefore specify this property explicitly for all quantities involved in the following considerations. Then we start our considerations at the level of the covariant Lagrangian for a charged massive particle. We think that a generally accepted version is, apart from sign conventions, given by

$$\mathscr{L} = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \Big(mc\sqrt{\dot{x}^\mu(\tau) g_{\mu\nu}(x(\tau)) \dot{x}^\nu(\tau)} + q\,A_\mu(x(\tau))\dot{x}^\mu(\tau)\Big) \qquad (1)$$

where, by definition, we have m is the invariant, constant (rest) mass; q is the invariant, constant charge; c is the invariant, constant velocity of light; τ is an invariant parameter (the proper time); x^μ are the coordinates of the four-dimensional spacetime, $\mu = 0, 1, 2, 3$; $x^\mu(\tau)$ is the position of the particle, parametrized by τ; $\dot{x}^\mu(\tau) = (\mathrm{d}/\mathrm{d}\tau)x^\mu(\tau)$ the covariant 4-velocity of the particle; $A_\mu(x)$ is the covariant, spacetime-dependent 4-potential of the electromagnetic field; $g_{\mu\nu}(x)$ is the second rank covariant tensor of the metric or gravitational field; we denote by $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ the Minkowskian metric.

We employ the concept of a test particle: the external fields are assumed to be given independently of the motion of the particle considered. As we do not consider the generation of the gravitational field we do

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not have to specify whether spacetime is curved or flat, this has an influence only on the admissible set of coordinate transformations. The equations of motion follow from (1) by Hamilton's variational principle, they are the Euler–Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta x^\mu(\tau)} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\delta \mathcal{L}}{\delta \dot{x}^\mu(\tau)} = 0. \tag{2}$$

These are four equations for the components of the covariant vector $\delta\mathcal{L}/\delta\dot{x}^\mu(\tau)$. In order that the parameter τ indeed is the proper time we have to restrict the solutions of (2) to those paths with

$$\dot{x}^\mu(\tau) g_{\mu\nu}(x(\tau))\dot{x}^\nu(\tau) = c^2. \tag{3}$$

We note that both quantities are scalars, the constancy of the LHS of (3) follows from (2). This restriction may be built into the Lagrangian (see e.g. Doughty (1990) for a modern treatment).

3. Two definitions of the energy

We note that the (rest-)mass is introduced as a constant whereas the energy so far is not part of this description. Thus we have to add a definition of the energy in terms of the quantities already defined. Now it is the essential point of this paper that there are two candidates for this definition. The conventional definition of the 4-momentum, of which E/c is to be the zeroth component, is

$$p^\mu(\tau) = m\dot{x}^\mu(\tau) \tag{4a}$$

which defines a contravariant vector. On the other hand there is the definition of the so-called 'generalized' momentum

$$P_\mu(\tau) = \frac{\delta\mathcal{L}}{\delta\dot{x}^\mu(\tau)} \tag{4b}$$

that defines a covariant vector. It is now our point that we want to argue that in the general context including a gravitational field only the quantity $P_0 c$ may justly be called the energy of the particle.

4. Energy conservation and Newtonian approximation

To see the difference between p^0 and P_0 we need the explicit equations of motion. From its definition we have

$$P_\mu = m g_{\mu\nu}\dot{x}^\nu + qA_\mu. \tag{5}$$

The Euler–Lagrange equations read, using (3)

$$\dot{P}_\mu = \tfrac{1}{2} m\dot{x}^\kappa\dot{x}^\lambda \frac{\partial}{\partial x^\mu} g_{\kappa\lambda} + q\dot{x}^\kappa \frac{\partial}{\partial x^\mu} A_\kappa. \tag{6}$$

Equations (5) and (6) may be combined to yield the more familiar equation

$$\dot{p}^\mu = -m\Gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda + qg^{\mu\kappa}F_{\kappa\lambda}\dot{x}^\lambda \tag{7}$$

with the Christoffel symbols $\Gamma^\mu_{\kappa\lambda}$ and the electromagnetic field tensor $F_{\kappa\lambda}$. We now see that whenever the external fields are time independent the quantity P_0 is conserved—this immediately follows from (2)—whereas p^0 is not. Since energy conservation is at the basis of all theoretical physics we may only say from $P_0 c$ that it is the conserved (total) energy of the particle. This becomes even clearer in a non-relativistic approximation of P_0 and p^0. We introduce the non-covariant coordinate velocity

$$v^i(t) = \frac{\mathrm{d}}{\mathrm{d}t} x^i(\tau(t)) = \gamma^1 \frac{\mathrm{d}}{\mathrm{d}\tau} x^i(\tau) \qquad v^0 = c \tag{8}$$

with

$$\gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau}(t) = \frac{c}{\sqrt{v^\mu g_{\mu\nu} v^\nu}} \tag{9}$$

which follows from (3). In a weak gravitational field with potential Φ the tensor $g_{\mu\nu}$ is specified by $g_{00} = 1 + 2\Phi/c^2$, $g_{ii} = -1$, we then have

$$P_0 c = \frac{mc^2 g_{00}}{\sqrt{g_{00} - v^2/c^2}} + qA_0 c \tag{10a}$$

and

$$p^0 c = m\gamma c^2 = \frac{mc^2}{\sqrt{g_{00} - v^2/c^2}}. \tag{10b}$$

This leads to the non-relativistic approximations

$$P_0 c \approx mc^2 + \tfrac{1}{2}mv^2 + m\Phi + qA_0 c \tag{11a}$$

and

$$p^0 c \approx mc^2 + \tfrac{1}{2}mv^2 - m\Phi. \tag{11b}$$

This shows that the non-relativistic limit of $P_0 c$ indeed is the total non-relativistic energy of the particle, including rest energy, kinetic and potential energy, whereas $P_0 c$ is the difference between the rest and kinetic energy and the potential energy in the gravitational field, hence it should not be called 'energy', which is obviously misleading since it has no particular meaning in the concept of energy. We moreover see that there is only one 'energy' in the general relativistic context, given by $P_0 c$, since this cannot be split into a kinetic and potential part in the presence of a gravitational field. These different terms belong to the Newtonian theory.

5. Conclusion

Following Einstein (1969) himself and many other authors (Adler 1987, Fadner 1988, Okun 1989) we conclude that it is the contribution of the rest energy mc^2 to the total energy that is the essence of Einstein's $E_0 = mc^2$. But from general principles it follows that neither the scalar mc^2 nor the contravariant $m\gamma c^2$ in general are identical with the energy of a massive charged particle in external gravitational and electromagnetic fields.

References

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